Topology Vol. 23, No. 1, pp. 101–115, 1984
Printed in Great Britain.
0040–9383/84 $3.00 + .00

THE STRATUM WITH CONSTANT MILNOR NUMBER OF A MINI-TRANSVERSAL FAMILY OF A QUASIHOMOGENEOUS FUNCTION OF CORANK TWO†

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(*Received* 18 *March* 1982)

§1. INTRODUCTION

LET $f:(\mathbf{C}^n, 0)\to(\mathbf{C}, 0)$ be a germ of quasihomogeneous function of Milnor number μ with an isolated critical point. Let $F:(\mathbf{C}^n \times \mathbf{C}^{\mu-1}, 0)\to(\mathbf{C}, 0)$ be transversal to the orbit of f in the space of germs of holomorphic functions preserving the origin with an isolated critical point, where we have the orbits of the action of the group of germs of biholomorphic mappings preserving the origin. We call the family F_t a *mini-transversal family* of f and call the *stratum with constant Milnor number* of F_t the germ at the origin of the set of those values of parameters t for which F_t has an isolated critical point at the origin of the same Milnor number as f. In [1], Arnol'd showed that the stratum with constant Milnor number of a minitransversal family contains a germ of non-singular algebraic subset in $\mathbf{C}^{\mu-1}$ of dimension $m(f)$, where $m(f)$ is the number of generators of a monomial basis of the finitely dimensional $\mathbf{C}$-vector space $\mathbf{C}\{x_1, \ldots, x_n\}/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ above and on the Newton boundary of f. We call the number $m(f)$ the *inner modarity* of f (see [1]). In [1], he conjectured that these germs coincide and showed this conjecture for 0- and 1-modal quasihomogeneous functions (see [1, 2]). In [13], we showed it for 2-modal quasi-homogeneous functions (see [1, 2]). In [4], Gabrielov and Kushnirenko showed it for homogeneous functions, using the result of Lê Dũng Tráng and Saito[7]. The author does not know of any further work on this problem.

In this paper, we shall determine the stratum with constant Milnor number of a mini-transversal family of a quasihomogeneous function of corank two with an isolated critical point, to prove the conjecture of Arnol'd for them. The details of our results are the following

THEOREM. *Let f be a quasihomogeneous function of corank two with an isolated critical point. Then the stratum with constant Milnor number of a mini-transversal family of f equals a germ of non-singular algebraic subset in $\mathbf{C}^{\mu-1}$ of dimension $m(f)$, where $m(f)$ is the inner modality of f and μ is the Milnor number of f.*

COROLLARY. *The modality of a quasihomogeneous function of corank two with an isolated critical point equals its inner modality (see the definition of modality in Preliminaries in this paper).*

The author wishes to thank Prof. E. Yoshinaga for useful suggestions.

§2. PRELIMINARIES

Let $f:(\mathbf{C}^n, o)\to(\mathbf{C}, o)$ be a germ of holomorphic function with an isolated critical point. The *Milnor number* of f is defined by the dimension of $\mathbf{C}\{x_1, \ldots, x_n\}/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$

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as a finite-dimensional $\mathbf{C}$-vector space and it is denoted by $\mu(f)$. Let $g: (\mathbf{C}^n, o) \to (\mathbf{C}, o)$ be a germ of holomorphic function with an isolated critical point and let $V(f)$, $V(g)$ be the germs at the origin of the hypersurfaces in $\mathbf{C}^n$ defined by f, g. Then $V(f)$ and $V(g)$ have the *same topological type* if there exist balls B_ε, B_δ in $\mathbf{C}^n$ with centre o and a local homeomorphism f from B_ε onto B_δ such that $h(B_\varepsilon \cap V(f)) = B_\delta \cap V(g)$. We denote this situation by $(\mathbf{C}^n, V(f)) \simeq (\mathbf{C}^n, V(g))$. In [11], Teissier showed that if $V(f)$ and $V(g)$ have the same topological type, then $\mu(f) = \mu(g)$.

Let $F: \mathbf{C}^n \times \mathbf{R} \to \mathbf{C}$ be a function which is holomorphic on $x \in \mathbf{C}^n$ and smooth (i.e. $\mathbf{C}^\infty$) on $t \in \mathbf{R}$. Suppose that for any $t \in \mathbf{R}$, $f(o, t) = 0$ and the germ $f_t: (\mathbf{C}^n, o) \to (\mathbf{C}, o)$ has an isolated critical point. Then in [6], Lê Dũng Tráng and Ramanujan showed the following

THEOREM 1. *If $\mu(f_t)$ is independent of $t \in R$ and $n \neq 3$, then $(\mathbf{C}^n, V(f_t)) \simeq (\mathbf{C}^n, V(f_0))$ for any $t \in \mathbf{R}$.*

We recall the result of Brauner[3]. Suppose that $V(f)$ is a germ at the origin of irreducible plane curve with an isolated singular point and with order m. Since the tangent cone of $V(f)$ has only one direction, by a coordinate transformation we have

$$f(x, y) = y^m + \text{higher terms}.$$

Then by Weierstrass' preparation theorem, we have the following up to a unit of the local ring $\mathbf{C}\{x, y\}$

$$f(x, y) = y^m + \sum_{i=1}^{m} y^{m-i} \phi_i(x),$$

where $\phi_i(x) \in \mathbf{C}\{x\}$ ord $(\phi_i) > i$, $i = 1, \ldots, m$. Let $y(x)$ be one of the m roots of $f(x, y) = 0$. Then the Puiseux development of $y(x)$ is given by the following (see [9])

$$y(x) = \sum_{i=0}^{p_0} a_{0i} x^{n_0 + i} + \sum_{i=0}^{p_1} a_{1i} x^{(n_1 + i)/m_1} + \cdots + \sum_{i=0}^{\infty} a_{gi} x^{(n_g + i)/m_1 \ldots m_g},$$

where

$$p_i, n_i \in \mathbf{N} \quad i = 0, \ldots, g \quad m_i \in \mathbf{N} \quad i = 1, \ldots, g$$

$$(m_i, n_i) = 1 \quad 1 < m_i < n_i \quad i = 1, \ldots, g$$

$$\frac{n_{i-1}}{m_1 \ldots m_{i-1}} < \frac{n_i}{m_1 \ldots m_i} \quad i = 2, \ldots, g$$

$$\prod_{i=1}^{g} m_i = m.$$

If $V(f)$ has no singular point, then the Puiseux development of $y(x)$ is reduced to the first sum by regrouping the terms with integer exponents in it. The result of Brauner is the following (see [3, 9]).

THEOREM 2. *The local topological type of a germ $V(f)$ at the origin of irreducible plane curve with an isolated singular point is completely determined by the pairs of integers (m_i, n_i) $i = 1, \ldots, g$.*

We call the pairs of integers (m_i, n_i) $i = 1, \ldots, g$ the *Puiseux characteristic pairs* of $V(f)$ (briefly, the *Pusieux pairs*). From this theorem, we see that the order m of f is a topological invariant of germs of irreducible plane curves.

If $V(f)$ is a germ of reducible plane curve, then the following theorem is useful (see [5, 12, 14]).

THEOREM 3. *Let $V(f)$, $V(g)$ be germs at the origin of plane curves in $\mathbf{C}^2$. Let*

$$V(f) = V_1 \cup \ldots \cup V_l \qquad V(g) = V'_1 \cup \ldots \cup V'_m$$

be decompositions of the germs into irreducible components. Then $V(f)$ and $V(g)$ have the same topological type if and only if there exists a bijection $B:\{1, \ldots, l\} \to \{1, \ldots, m\}$ such that

$$(\mathbf{C}^2, V_i) \simeq (\mathbf{C}^2, V'_{B(i)})$$

$$(V_i \cdot V_j)_0 = (V'_{B(i)} \cdot V'_{B(j)})_0 \quad \textit{for any } i, j,$$

where $(\cdot)_0$ is the intersection number at the origin.

A germ of polynomial function $f:(\mathbf{C}^n, o) \to (\mathbf{C}, o)$ is a *quasihomogeneous function* of type $(d; r_1, \ldots, r_n)$ if

$$f(t^{r_1}x_1, \ldots, t^{r_n}x_n) = t^d f(x_1, \ldots, x_n) \quad \text{for any } t \in \mathbf{R},$$

where d, r_i $i = 1, \ldots, n$ are positive rational numbers. We call the number d the *degree* of f and call the number r_i $i = 1, \ldots, n$ the *weights* of f. Let $g:(\mathbf{C}^n, o) \to (\mathbf{C}, o)$ be a germ of polynomial and let $g = g_{d_1} + \cdots + g_{d_k}$ $(d_1 < \cdots < d_k)$ be the decomposition of g into quasihomogeneous functions of types $(d_i; r_1, \ldots, r_n)$ $i = 1, \ldots, k$. Then we use the following notation

$$\mathrm{In}(g) = g_{d_1}, \ \mathrm{ord}(g) = d_1, \ \deg(g) = d_k \quad \text{for } (r_1, \ldots, r_n).$$

The following result of K. Saito[10] is useful.

THEOREM 4. *Let $f(x)$ be a quasihomogeneous function of type $(d; r_1, \ldots, r_n)$ with an isolated critical point. Then there exists a coordinate transformation $x_i = \Phi_i(y)$ $i = 1, \ldots, n$ such that*

$$f(\Phi(y)) = h(y_1, \ldots, y_k) + y_{k+1}^2 + \cdots + y_n^2,$$

where h is a quasihomogeneous function of type $(1; s_1, \ldots, s_k)$ with $0 < s_i < 1/2$ $i = 1, \ldots, k$. And the type of h is uniquely determined.

Note that h has an isolated critical point. We call the quasihomogeneous function h the *principal part* of f and call the number k the *corank* of f. Mini-transversal families of f, h are analytically equivalent after adding non-degenerate quadratic form $y_{k+1}^2 + \cdots + y_n^2$. Hence in what follows, we shall consider a quasihomogeneous function of type $(1; r_1, \ldots, r_n)$ (briefly, $(r_1, \ldots, r_n)$) with $0 < r_i < 1/2$ $i = 1, \ldots, n$ and with an isolated critical point.

The *modality* of a germ of holomorphic function is defined by the largest number m such that there exists in its mini-transversal family an m-dimensional analytic subset whose intersection with every orbit of the group of germs of biholomorphic mappings preserving the origin is empty or discrete (see [2]).

§3. PROOF OF THEOREM

Let $f:(\mathbf{C}^n, o)\to(\mathbf{C}, o)$ be a germ of holomorphic function of Milnor number μ with an isolated critical point. We denote by $\mathscr{S}_F$ the stratum with constant Milnor number of a mini-transversal family F_t of f, namely as a germ at the origin

$$\mathscr{S}_F=\{t\in\mathbf{C}^{\mu-1}|\mu(F_t)=\mu(f)\}.$$

It is well known that the stratum $\mathscr{S}_F$ is a germ of algebraic subset at the origin in $\mathbf{C}^{\mu-1}$.

Let F_t be the mini-transversal family of f, given by

$$F(x,t)=f(x)+\sum_{i=1}^{\mu-1}t_i\phi_i(x),$$

where $\{\phi_1,\ldots,\phi_{\mu-1}\}$ is a monomial basis of a finite-dimensional $\mathbf{C}$-vector space $\mathscr{L}_f:=\mathscr{M}^2/\mathscr{M}(\partial f/\partial x_1,\ldots,\partial f/\partial x_n)$ ($\mathscr{M}$ is the maximal ideal of the local ring $\mathbf{C}\{x_1,\ldots,x_n\}$). If f is quasihomogeneous of type $(r_1,\ldots,r_n)$, then by Arnol'd[1], we see that

$$\mathscr{S}_F\supset\mathscr{A}_F,$$

where $\mathscr{A}_F$ is the germ at the origin of the set

$$\{(t_1,\ldots,t_{\mu-1})\in\mathbf{C}^{\mu-1}|t_i=0\text{ for }i\text{ for which }\deg(\phi_i)<1\text{ for }(r_1,\ldots,r_n)\}.$$

In what follows, we will denote representatives of $\mathscr{S}_F$, $\mathscr{A}_F$ be the same symbols whenever this is not confusing.

We shall consider the following quasihomogeneous functions of three types, given by

$$\begin{aligned}
f_1(x,y)&=x^a+y^b+g(x,y) && a,b\geqq 3\\
f_2(x,y)&=x(x^a+y^b+g(x,y)) && a,b\geqq 2\\
f_3(x,y)&=xy(x^a+y^b+g(x,y)) && a,b\geqq 1,
\end{aligned}$$

where g is a quasihomogeneous function of type $(1/a, 1/b)$ and g does not contain the monomials x^a, y^b with non-zero coefficients and f_i $(i=1,2,3)$ has an isolated critical point. Note that f_1, f_2, f_3 are quasihomogeneous of type $(1/a,\ 1/b)$, $(1/(a+1),\ a/(a+1)b)$, $(b/(ab+a+b),\ a/(ab+a+b))$ respectively.

Let k be the greatest common measure of the exponents a, b and put $a=ck$, $b=dk$. Then we see that

$$f_1(x,y)=x^a+y^b+g(x,y)=(x^c)^k+(y^d)^k+\sum_{i+j=k}a_{ij}(x^c)^i(y^d)^j.$$

Let $\zeta_i\in\mathbf{C}^*$ $(i=1,\ldots,k)$ be the roots of the equation

$$1+X^k+\sum_{i+j=k}a_{ij}X^j=0.$$

Then we have the decomposition of $f_1(x,y)$ in the local ring $\mathbf{C}\{x,y\}$ into irreducible components

$$f_1(x,y)=x^a+y^b+g(x,y)=\prod_{i=1}^{k}(y^d+\zeta_i x^c).$$

Note that $\zeta_i\neq\zeta_j$ for any i,j $(i\neq j)$ since $f_1(x,y)$ has an isolated critical point and it is square free.

Note that for any i $(i\geqq 1)$

$$bx^iy^{b-1}+x^i\frac{\partial g}{\partial y}(x,y)=x^i\frac{\partial f_1}{\partial y}(x,y).$$

Thus we can find the mini-transversal family F_{1t} of f_1, given by the following

$$F_1(x,y,t)=f_1(x,y)+\sum_{i=1}^{\mu_1-1}t_i\phi_{1i}(x,y),$$

where $\{\phi_{11},\dots,\phi_{1\mu_1-1}\}$ is a monomial basis of $\mathscr{L}_{f_1}$ which does not contain the monomials x^iy^{b-1} $(i\geqq 1)$ and μ_1 is the Milnor number of f_1. Note that for any i $(i\geqq 1)$

$$(a+1)x^ay^i+y^{b+i}+y^i(g(x,y)+x\frac{\partial g}{\partial x}(x,y))=y^i\frac{\partial f_2}{\partial x}(x,y)$$

$$bx^{i+1}y^{b-1}+x^{i+1}\frac{\partial g}{\partial y}(x,y)=x^i\frac{\partial f_2}{\partial y}(x,y).$$

Thus we can find the mini-transversal families F_{2t}, F'_{2t} of f_2, given by the following

$$F_2(x,y,t)=f_2(x,y)+\sum_{i=0}^{\mu_2-1}t_i\phi_{2i}(x,y)$$

$$F'_2(x,y,t)=f_2(x,y)+\sum_{i=0}^{\mu_2-1}t_i\phi'_{2i}(x,y),$$

where $\{\phi_{21},\dots,\phi_{2\mu_2-1}\}$(resp. $\{\phi'_{21},\dots,\phi'_{2\mu_2-1}\}$) is a monomial basis of $\mathscr{L}_{f_2}$ which does not contain the monomials y^{b+i} $(i\geqq 1)$, $x^{i+1}y^{b-1}$ $(i\geqq 1)$ (resp. x^ay^i $(i\geqq 1)$) and μ_2 is the Milnor number of f_2. Note that for any i $(i\geqq 1)$

$$y^{b+1+i}+(a+1)x^ay^{i+1}+y^{i+1}g(x,y)+xy^{i+1}\frac{\partial g}{\partial x}(x,y)=y^i\frac{\partial f_3}{\partial x}(x,y)$$

$$(b+1)x^{i+1}y^b+x^{a+1+i}+x^{i+1}g(x,y)+x^{i+1}y\frac{\partial g}{\partial y}(x,y)=x^i\frac{\partial f_3}{\partial y}(x,y).$$

Thus we have the mini-transversal family F_{3t} of f_3, given by the following

$$F_3(x,y,t)=f_3(x,y)+\sum_{i=0}^{\mu_3-1}t_i\phi_{3i}(x,y),$$

where $\{\phi_{31},\dots,\phi_{3\mu_3-1}\}$ is a monomial basis of $\mathscr{L}_{f_3}$ which does not contain the monomials $x^{i+1}y^b$ $(i\geqq 1)$, y^{b+1+i} $(i\geqq 1)$ and μ_3 is the Milnor number of f_3.

We shall determine the strata $\mathscr{S}_{F_i}$ $i = 1, 2, 3$ for the mini-transversal families F_i $i = 1, 2, 3$.

LEMMA 1. *For the mini-transversal family F_{1t} of the quasihomogeneous function f_1, we have*

$$\mathscr{S}_{F_1} = \mathscr{A}_{F_1}.$$

Proof. Suppose that the germ of the semi-algebraic subset $\mathscr{S}_{F_1} - \mathscr{A}_{F_1}$ at the origin in $\mathbf{C}^{\mu_1 - 1}$ is not empty; so that the topological closure of $\mathscr{S}_{F_1} - \mathscr{A}_{F_1}$ contains the origin. By the curve selection lemma (see [8]), we can find a real analytic curve $\lambda:[0, \infty)\to\mathbf{C}^{\mu_1 - 1}$ with $\lambda(0) = 0$ and with $\lambda(t) \in \mathscr{S}_{F_1} - \mathscr{A}_{F_1} \neq \phi$ for any $t > 0$. As noted before, there exist non-zero complex numbers $\zeta_i \in C$ $i = 1, \ldots, k$ ($\zeta_i \neq \zeta_j$ $i \neq j$) for which we have the decomposition of f_1 in the local ring $\mathbf{C}\{x, y\}$ into irreducible components

$$f_1(x, y) = \prod_{i=1}^{k} (y^d + \zeta_i x^c) := \prod_{i=1}^{k} f_{1i}(x, y).$$

Since $F_{1\lambda(t)}$ is a smooth family on $t \in [0, \infty)$ with a constant Milnor number $\mu(f_1)$, we have by Theorem 1

$$(\mathbf{C}^2, V(F_{1\lambda(t)})) \simeq (\mathbf{C}^2, V(f_1)) \quad \text{for any } t \in [0, \infty).$$

Hence, if $F_{1\lambda(t)}$ has the irreducible decomposition in $\mathbf{C}\{x, y\}$

$$F_{1\lambda(t)} = \prod_{i=1}^{k} f_{it} \quad (\text{for any } t \in [0, \infty)),$$

then, by Theorem 3 we have, for any $i, j = 1, \ldots, k$, $i \neq j$ and any $t \in [0, \infty)$

$$(\mathbf{C}^2, V(f_{1i})) \simeq (\mathbf{C}^2, V(f_{it}))$$

$$(V(f_{1i}) \cdot V(f_{1j}))_0 \simeq (V(f_{it}) \cdot V(f_{jt}))_0.$$

Note that f_{it} ($i = 1, \ldots, k$) is at least continuous in the real parameter t since there exists a simultaneous stable radius for the family $V(F_{1\lambda(t)})$ of germs of plane curves with constant Milnor number (see [6, 15]).

Without loss of generality, we may assume that $b \leqq a$. In what follows, any, hereafter fixed, sufficiently small parameter $t > 0$ is chosen.

If f_{1i} ($i = 1, \ldots, k$) has an isolated critical point, i.e. $d \geqq 2$, then $V(f_{1i})$ ($i = 1, \ldots, k$) is a germ of irreducible plane curve with the Puiseux pair (d, c). Since we have

$$(\mathbf{C}^2, V(f_{it})) \simeq (\mathbf{C}^2, V(f_{1i})) \quad i = 1, \ldots, k,$$

if $d \geqq 2$, then the Puiseux pair of the germ of the irreducible plane curve $V(f_{it})$ ($i = 1, \ldots, k$) is equal to (d, c) and the order of f_{it} ($i = 1, \ldots, k$) is equal to d. Note that the order of f_{it} ($i = 1, \ldots, k$) is equal to d even if f_{it} ($i = 1, \ldots, k$) has no critical point. Since the tangent cone of $V(f_{it})$ ($i = 1, \ldots, k$) has only one direction, we have for any i ($i = 1, \ldots, k$)

$$f_{it}(x, y) = (b_{0i}y + a_{0i}x)^d + \text{higher terms} \quad b_{0i} \neq 0.$$

where $b_{0i} \neq 0$ because that f_{it} is close to f_{1i} for any i $(i = 1, \dots, k)$. Then by Weierstrass' preparation theorem,

$$f_{it}(x, y) = u_{it}(x, y)\left\{(y + a'_{0i}x)^d + \sum_{j=1}^{d} (y + a'_{0i}x)^{d-j}\phi_{ij}(x)\right\} \quad i = 1, \dots, k,$$

where u_{it} $(i = 1, \dots, k)$ is a unit of the local ring $\mathbf{C}\{x, y\}$ and

$$\phi_{ij} \in \mathbf{C}\{x\} \quad \operatorname{ord}(\phi_{ij}) > j \quad \text{for any } i, j \ (i = 1, \dots, k \quad j = 1, \dots, d).$$

We redefine

$$f_{it}(x, y) = (y + a'_{0i}x)^d + \sum_{j=1}^{d} (y + a'_{0i}x)^{d-j}\phi_{ij}(x) \quad i = 1, \dots, k$$

and put

$$u_t = \prod_{i=1}^{k} u_{it}.$$

Let $y_i(x)$ $(i = 1, \dots, k)$ be one of the d roots of the equation $f_{it}(x, y) = 0$. Then the Puiseux development of $y_i(x)$ is given by the following

$$y_i(x) = \sum_{j=0}^{p_i} a_{ij}x^{c_i+j} + \sum_{j=0}^{\infty} b_{ij}x^{(c+j)/d} \quad b_{i0} \neq 0 \quad i = 1, \dots, k,$$

where $c_i \in \mathbf{N}$ $1 \leqq c_i + j < c/d$ $a_{i0} \neq 0$ $i = 1, \dots, k$ $j = 0, \dots, p_i$ if the first sum in the above expansion is non-zero. If f_{1i} $(i = 1, \dots, k)$ has no critical point, i.e. $d = 1$, then the terms in the second sum of the Puiseux development $y_i(x)$ have integer exponents on x. Put

$$X = x, \quad Y_i = y - \sum_{j=0}^{p_i} a_{ij}x^{c_i+j} \quad i = 1, \dots, k.$$

Then, in terms of X, Y_i, we can write

$$\begin{aligned} f_{it}(x, y) &= \prod_{\eta^d = 1} (y - y_i(\eta \cdot x)) \\ &= \prod_{\eta^d = 1} (Y_i - Y_i(\eta \cdot X)) \\ &= Y_i^d + (-b_{i0})^d X^c + \psi_{it}(X, Y_i) \quad i = 1, \dots, k, \end{aligned}$$

where

$$y_i(\eta \cdot x) = \sum_{j=0}^{p_i} a_{ij}x^{c_i+j} + \sum_{j=0}^{\infty} b_{ij}\eta^{c+j}x^{(c+j)/d}$$

$$Y_i(\eta \cdot X) = \sum_{j=0}^{\infty} b_{ij}\eta^{c+j}X^{(c+j)/d}$$

$$\psi_{it} \in \mathbf{C}\{X, Y_i\} \quad \operatorname{ord}(\psi_{it}) > 1 \quad \text{for } (1/c, 1/d) \quad i = 1, \dots, k.$$

Thus we have for any i $(i = 1, \ldots, k)$

$$f_{it}(x, y) = \left(y - \sum_{j=0}^{p_i} a_{ij} x^{c_i + j}\right)^d + (-b_{i0})^d x^c + \psi_{ij}\left(x, y - \sum_{j=0}^{p_i} a_{ij} x^{c_i + j}\right)$$

and if the first sum in the development of $y_i(x)$ is non-zero, then we have

$$\mathrm{In}\,(f_{it}) = (y - a_{i0} x^{c_i})^d \quad \text{for } (1/c_i d, 1/d)$$

and if not, then we have

$$\mathrm{In}\,(f_{it}) = y^d + (-b_{i0})^d x^c \quad \text{for } (1/c, 1/d).$$

Note that $\mathrm{In}\,(u_t) =$ *a constant* for any weights. By the hypothesis that $\lambda(t) \in \mathscr{S}_{F_1} - \mathscr{A}_{F_1} \neq \phi$ for $t > 0$, there exists i for which the Puiseux expansion of $y_i(x)$ has non-zero first sum. Because if not, then we have

$$\mathrm{ord}\,(F_{1\lambda(t)}) = \mathrm{ord}\left(u_t \cdot \prod_{i=1}^{k} f_{it}\right) \geqq \deg\,(f_1) = k \quad \text{for } (1/c, 1/d)$$

namely

$$\mathrm{ord}\,(F_{1\lambda(t)}) \geqq \deg\,(f_1) = 1 \quad \text{for } (1/a, 1/b).$$

Thus we assume c_1 to be the minimum of the c_i's for i's where $y_i(x)$ has Puiseux expansion with non-zero first sum. Suppose that $a_{i0} = a_{10}$ if $c_i = c_1$. Let m be the number of the set $\{i \mid c_i = c_1\}$. Then we see that

$$\mathrm{In}\,(F_{1\lambda(t)}) = K \cdot y^{(k-m)d}(y - a_{10} x^{c_1})^{md} \quad \text{for } (1/c_1 d, 1/d)$$

since

$$\begin{cases} \mathrm{In}\,(f_{it}) = (y - a_{10} x^{c_1})^d & \text{for } (1/c_1 d, 1/d) \text{ if } c_i = c_1 \\ \mathrm{In}\,(f_{it}) = y^d & \text{for } (1/c_1 d, 1/d) \text{ otherwise} \end{cases}$$

$$\mathrm{In}\,(u_t) = K \text{ (a constant)} \quad \text{for any weights.}$$

Thus $F_{1\lambda(t)}$ contains the monomial $y^{b-1} x^{c_1}$ with non-zero coefficient. But this contradicts the fact the $F_{1\lambda(t)}$ does not contain the monomials $x^i y^{b-1}$ $(i \geqq 1)$ with non-zero coefficient. Hence if $m = 1$, we have $\mathscr{S}_{F_1} - \mathscr{A}_{F_1} = \phi$, and $\mathscr{S}_{F_1} = \mathscr{A}_{F_1}$ since $\mathscr{S}_{F_1} \supset \mathscr{A}_{F_1}$. Note that $m = 1$ if f_1 is irreducible, i.e. $k = 1$. If $m \geqq 2$, then we can find some $l (\neq 1)$ for which $c_l = c_1$ and $a_{l0} \neq a_{10}$. Then we see that

$$\begin{aligned} cd &= (V(f_{11}) \cdot V(f_{1l}))_0 \\ &= (V(f_{1t}) \cdot V(f_{lt}))_0 \\ &= \mathrm{ord}\,(f_{lt}(\tau^d, y_1(\tau^d))) \\ &= \mathrm{ord}\left(\prod_{\eta^d = 1} (y_1(\tau^d) - y(\eta \cdot \tau^d)\right) \end{aligned}$$

$$= \operatorname{ord}((a_{10} - a_0)^d \cdot \tau^{c_1 d^2} + \text{higher terms})$$

$$= c_1 d^2.$$

But this contradicts $c_1 < c/d$. Hence we have $\mathscr{S}_{F_1} - \mathscr{A}_{F_1} = \phi$, and we have $\mathscr{S}_{F_1} = \mathscr{A}_{F_1}$ since $\mathscr{S}_{F_1} \supset \mathscr{A}_{F_1}$. These complete the proof.

LEMMA 2. *If $b \leqq a$, then for the mini-transversal family F_{2t} of the quasihomogeneous function f_2, we have*

$$\mathscr{S}_{F_2} = \mathscr{A}_{F_2}.$$

If $a \leqq b$, then for the mini-transversal family F'_{2t} of the quasihomogeneous function f_2, we have

$$\mathscr{S}_{F'_2} = \mathscr{A}_{F'_2}.$$

Proof. Suppose that the germ of the semi-algebraic subset $\mathscr{S}_{F_2} - \mathscr{A}_{F_2}$(resp. $\mathscr{S}_{F'_2} - \mathscr{A}_{F'_2}$) at the origin in $\mathbf{C}^{\mu_2 - 1}$ is not empty. Then we can find a real analytic curve $\lambda : [0, \infty) \to \mathbf{C}^{\mu_2 - 1}$ with $\lambda(0) = 0$ and with $\lambda(t) \in \mathscr{S}_{F_2} - \mathscr{A}_{F_2}$ (resp. $\mathscr{S}_{F'_2} - \mathscr{A}_{F'_2}$) for any $t > 0$. Corresponding to the decomposition of f_2 in $\mathbf{C}\{x, y\}$ into irreducible components

$$f_2(x, y) = x \cdot \prod_{i=0}^{k} (y^d + \zeta_i x^c) := \prod_{i=0}^{k} f_{2i}(x, y),$$

we have the decomposition of $F_{2\lambda(t)}$(resp. $F'_{2\lambda(t)}$) in $\mathbf{C}\{x, y\}$ into irreducible components

$$F_{2\lambda(t)} = \prod_{i=0}^{k} f_{it} \left(\text{resp. } F'_{2\lambda(t)} = \prod_{i=0}^{k} f'_{it} \right) \text{ for any } t \in [0, \infty).$$

Hence, by Theorems 1 and 3 we have for any i, j $(i, j = 0, \ldots, k \; i \neq j)$ and any $t \in [0, \infty)$

$$(\mathbf{C}^2, V(f_{2i})) \simeq (\mathbf{C}^2, V(f_{it})) \text{ (resp. } (\mathbf{C}^2, V(f'_{2i})) \simeq (\mathbf{C}^2, V(f'_{it})))$$

$$(V(f_{2i}) \cdot V(f_{2j}))_0 = (V(f_{it}) \cdot V(f_{jt}))_0 \text{(resp. } (V(f'_{2i}) \cdot V(f'_{2j}))_0 = (V(f'_{it}) \cdot V(f'_{jt}))_0).$$

In what follows, any, hereafter fixed, sufficiently small parameter $t > 0$ is chosen.

First we shall consider the case where $b \leqq a$. In the same way as in Lemma 1, we have the following up to a unit $u_{it} \in \mathbf{C}\{x, y\}$ $(i = 1, \ldots, k)$

$$f_{it}(x, y) = \left(y - \sum_{j=0}^{p_i} a_{ij} x^{c_i + j} \right)^d + (-b_{i0})^d x^c + \psi_{ij} \left(x, y - \sum_{j=0}^{p_i} a_{ij} x^{c_i + j} \right) \quad i = 1, \ldots, k.$$

Since $F_{2\lambda(t)}$ does not contain the monomials y^{b+i} $(i \geqq 1)$ with non-zero coefficients, we have $f_{0t}(o, y) \equiv 0$. Thus, since f_{0t} is close to $f_{20} = x$, we have

$$f_{0t}(x, y) = u_{0t} \cdot x,$$

where u_{0t} is a unit of the local ring $\mathbf{C}\{x, y\}$. We redefine

$$f_{0t}(x, y) = x$$

and put

$$u_t = \prod_{i=0}^{k} u_{it}.$$

By the hypothesis that $\lambda(t) \in \mathscr{S}_{F_2} - \mathscr{A}_{F_2} \neq \phi$ for $t > 0$, we can find the exponent c_1 as in Lemma 1. Because if not, then we have

$$\operatorname{ord}(F_{2\lambda(t)}) = \operatorname{ord}\left(u_t \cdot \prod_{i=0}^{k} f_{it}\right) \geqq \deg(x) + \sum_{i=1}^{k} \deg(f_{it})$$

$$= 1/c + k \text{ for } (1/c, 1/d)$$

namely

$$\operatorname{ord}(F_{2\lambda(t)}) \geqq 1 \text{ for } (1/(a+1),\ a/(a+1)b).$$

Suppose that $a_{i0} = a_{10}$ if $c_i = c_1$ and let m be the number of the set $\{i | c_i = c_1\}$. Then $F_{2\lambda(t)}$ contains the terms $K \cdot xy^{(k-m)d}(y - a_{10}x^{c_1})^{md}$ since

$$\begin{cases} \operatorname{In}(f_{it}) = (y - a_{10}x^{c_1})^d & \text{for } (1/c_1 d, 1/d) \text{ if } c_i = c_1 \\ \operatorname{In}(f_{it}) = y^d & \text{for } (1/c_1 d, 1/d) \text{ otherwise} \end{cases}$$

$$\operatorname{In}(u_t) = K \text{ (a constant) for any weights.}$$

Thus $F_{2\lambda(t)}$ contains the monomial $x^{c_1+1}y^{b-1}$ with non-zero coefficient. But this contradicts the fact that $F_{2\lambda(t)}$ does not contain the monomials $x^{i+1}y^{b-1}$ $(i \geqq 1)$ with non-zero coefficients. Hence if $m = 1$, then we have $\mathscr{S}_{F_2} - \mathscr{A}_{F_2} = \phi$, and we have $\mathscr{S}_{F_2} = \mathscr{A}_{F_2}$ since $S_{F_2} \supset \mathscr{A}_{F_2}$. If $m \geqq 2$, then there exist some $l (\neq 1)$ for which $c_l = c_1$ and $a_{l0} = a_{10}$. By the same discussion of the intersection number as in Lemma 1, we have $c_1 = c/d$, but this is a contradiction. Thus we have $\mathscr{S}_{F_2} - \mathscr{A}_{F_2} = \phi$, and we have $\mathscr{S}_{F_2} = \mathscr{A}_{F_2}$ since $\mathscr{S}_{F_2} \supset \mathscr{A}_{F_2}$. Hence we have the contradiction in the case where $b \leqq a$.

Second we shall consider the case where $a \leqq b$. Then it is enough to consider

$$f_2(x, y) = yf_1(x, y) \quad b \leqq a.$$

We consider the mini-transversal family F'_{2t} corresponding to the new f_2. Then note that F'_{2t} does not contain the monomials $x^i y^b$ $(i \geqq 1)$. Since f'_{0t} is close to f_{20}, we have

$$f'_{0t}(x, y) = b'_{00}y + a'_{00}x + \text{higher terms} \quad b'_{00} \neq 0.$$

By Weierstrass' preparation theorem, we have

$$f'_{0t}(x, y) = u_{0t}(x, y)\left\{y - \sum_{j=0}^{p_0} a_{0j}x^{c_0+j} - \sum_{j=0}^{\infty} b_{0j}x^{d_0+j}\right\} \quad d_0 \in N \quad d_0 \geqq c/d,$$

where $c_0 \in \mathbf{N}$ $1 \leqq c_0 + j < c/d$ $a_{00} \neq 0$ if the first sum in the above expansion is non-zero. In the same way as in Lemma 1, we have the following up to a unit $u_{it} \in \mathbf{C}\{x, y\}$ $(i = 1, \ldots, k)$

$$f'_{it}(x, y) = \left(y - \sum_{j=0}^{p_i} a_{ij}x^{c_i+j}\right)^d + (-b_{i0})^d x^c + \psi_{ij}\left(x, y - \sum_{j=0}^{p_i} a_{ij}x^{c_i+j}\right)$$

for $i = 1, \ldots, k$. We redefine

$$f'_{0t}(x, y) = y - \sum_{j=0}^{p_0} a_{0j}x^{c_0+j} - \sum_{j=0}^{\infty} b_{0j}x^{d_0+j}$$

and put

$$u_{0t} = \prod_{i=0}^{k} u_{it}.$$

By the hypothesis that $\lambda(t) \in \mathscr{S}_{F'_2} - \mathscr{A}_{F'_2} = \phi$ for $t > 0$, there exist some $i (0 \leqq i \leqq k)$ for which $a_{i0} \neq 0$. Because if not, then we have

$$\operatorname{ord}(F'_{2\lambda(t)}) = \operatorname{ord}\left(u_t \cdot \prod_{i=0}^{k} f'_{it}\right) \geqq \deg(y) + \sum_{i=1}^{k} \deg(f'_{it})$$

$$= 1/d + k \text{ for } (1/c, 1/d)$$

namely

$$\operatorname{ord}(F'_{2\lambda(t)}) \geqq 1 \text{ for } (b/(b+1)a, 1/(b+1)).$$

Note that f_2 quasihomogeneous of type $(b/(b+1)a, 1/(b+1))$. Let c_g be the minimum of the set $\{c_i | a_{i0} \neq 0 \ (0 \leqq i \leqq k)\}$. Suppose that $a_{g0} = a_{i0}$ if $c_g = c_i$. Then if $c_g = c_0$, then $F'_{2\lambda(t)}$ contains the terms

$$K \cdot y^{(k-m)d}(y - a_{00}x^{c_g})^{md+1}$$

and if not, then $F'_{2\lambda(t)}$ contains the terms

$$K \cdot y^{(k-m)d+1}(y - a_{g0}x^{c_g})^{md}$$

since

$$\begin{cases} \operatorname{In}(f'_{it}) = (y - a_{g0}x^{c_g})^d & \text{for } (1/c_g d, 1/d) \text{ if } c_i = c_g \\ \operatorname{In}(f'_{it}) = y^d & \text{for } (1/c_g d, 1/d) \text{ otherwise} \end{cases}$$

$$\begin{cases} \operatorname{In}(f'_{0t}) = y - a_{00}x^{c_0} & \text{for } (1/c_0 d, 1/d) \text{ if } c_0 = c_g \\ \operatorname{In}(f'_{0t}) = y & \text{for } (1/c_0 d, 1/d) \text{ otherwise} \end{cases}$$

$$\operatorname{In}(u_t) = K \text{ (a constant) for any weights.}$$

Thus in both cases, $F'_{2\lambda(t)}$ contains the monomial $x^{c_g}y^b$ with non-zero coefficient. But this contradicts the fact that $F'_{2\lambda(t)}$ does not contain the monomials $x^i y^b$ $(i \geqq 1)$ with non-zero coefficients. Hence if $\# \{i | c_g = c_i \ (0 \leqq i \leqq k)\} = 1$, then we have $\mathscr{S}_{F'_2} - \mathscr{A}_{F'_2} = \phi$, and we have $\mathscr{S}_{F'_2} = \mathscr{A}_{F'_2}$ since $\mathscr{S}_{F'_2} \supset \mathscr{A}_{F'_2}$. If $\# \{i | c_g = c_i \ (0 \leqq i \leqq k)\} \geqq 2$, then there exist some $l(\neq g)$ for which $c_g = c_l$ and $a_{g0} = a_{l0}$. Then if $g = 0$, we have

$$
\begin{aligned}
c &= (V(f'_{20}) \cdot V(f'_{2l}))_0 \\
&= (V(f'_{0t}) \cdot V(f'_{lt}))_0 \\
&= \operatorname{ord}\,(f'_{lt}(\tau, y_0(\tau))) \\
&= \operatorname{ord}\left(\prod_{\eta^d=1} (y_0(\tau) - y_l(\eta \cdot \tau))\right) \\
&= \operatorname{ord}\,((a_{00} - a_{l0})^d \cdot \tau^{c_0 d} + \text{higher terms}) \\
&= c_0 d,
\end{aligned}
$$

where

$$
y_0(x) = \sum_{j=0}^{p_0} a_{0j} x^{c_0 + j} + \sum_{j=0}^{\infty} b_{0j} x^{d_0 + j}
$$

and $y_l(x)$ has the same Puiseux development as in Lemma 1. But this contradicts $c_0 < c/d$. Next if $g, l \neq 0$, then we have $c_0 = c/d$ in the same discussion of the intersection number as in Lemma 1. Thus we have $\mathscr{S}_{F'_2} - \mathscr{A}_{F'_2} = \phi$, and we have $\mathscr{S}_{F'_2} = \mathscr{A}_{F'_2}$ since $\mathscr{S}_{F'_2} \supset \mathscr{A}_{F'_2}$. Hence we have the conclusion in the case where $a \leqq b$. These complete the proof.

LEMMA 3. *For the mini-transversal family F_{3t} of the quasihomogeneous function f_3, we have*

$$
\mathscr{S}_{F_3} = \mathscr{A}_{F_3}.
$$

Proof. Suppose that the germ of the semi-algebraic subset $\mathscr{S}_{F_3} - \mathscr{A}_{F_3}$ at the origin in $\mathbf{C}^{\mu_3 - 1}$ is not empty. Then we can choose a real analytic curve $\lambda : [0, \infty) \to \mathbf{C}^{\mu_3 - 1}$ with $\lambda(0) = 0$ and with $\lambda(t) \in \mathscr{S}_{F_3} - \mathscr{A}_{F_3}$ for $t > 0$. Corresponding to the decomposition of f_3 in $\mathbf{C}\{x, y\}$ into irreducible components

$$
f_3(x, y) = xy \cdot \prod_{i=1}^{k} (y^d + \zeta_i x^c) := \prod_{i=-1}^{k} f_{3i}(x, y),
$$

we have the decomposition of $F_{2\lambda(t)}$ in $\mathbf{C}\{x, y\}$ into irreducible components

$$
f_3(x, y) = xy \prod_{i=-1}^{k} (y^d + \zeta_i x^c) := \prod_{i=-1}^{k} F_{3i}(x, y),
$$

we have the decomposition of $F_{2\lambda(t)}$ in $\mathbf{C}\{x, y\}$ into irreducible components

$$
F_{3\lambda(t)} = \prod_{i=-1}^{k} f_{it} \text{ for any } t \in [0, \infty).
$$

Hence, by Theorems 1 and 3 we have for any i, j $(i, j = -1, \ldots, k\ i \neq j)$ and any $t \in [0, \infty)$

$$
(\mathbf{C}^2, V(f_{3i})) \simeq (\mathbf{C}^2, V(f_{it}))
$$

$$
(V(f_{3i}) \cdot V(f_{3j}))_0 = (V(f_{it}) \cdot V(f_{jt}))_0.
$$

In what follows, any, hereafter fixed, sufficiently small parameter $t > 0$ is chosen. Without loss of generality, we may assume that $b \leqq a$.

In the same way as in the preceeding lemma, we have the following up to a unit $u_{it} \in \mathbf{C}\{x, y\}$ $(i = 0, \ldots, k)$

$$f_{0t}(x, y) = y - \sum_{j=0}^{p_0} a_{0j} x^{c_0 + j} - \sum_{j=0}^{\infty} b_{0j} x^{d_0 + j}$$

$$f_{it}(x, y) = \left(y - \sum_{j=0}^{p_0} a_{ij} x^{c_i + j}\right)^d + (-b_{i0})^d x^c + \psi_{ij}\left(x, y - \sum_{j=0}^{p_i} a_{ij} x^{c_i + j}\right)$$

for any i $(i = 1, \ldots, k)$. Since $F_{3\lambda(t)}$ does not contain the monomials y^{b+1+i} $(i \geqq 1)$ with non-zero coefficients, we have $f_{-1t}(0, y) \equiv 0$. Thus, since f_{-1t} is close to $f_{3,-1} = x$, we have

$$f_{-1t}(x, y) = u_{-1t} \cdot x,$$

where u_{-1t} is a unit of the local ring $\mathbf{C}\{x, y\}$. We redefined

$$f_{-1t}(x, y) = x$$

and put

$$u_t = \prod_{i=-1}^{k} u_{it}.$$

By the hypothesis that $\lambda(t) \in \mathscr{S}_{F_3} - \mathscr{A}_{F_3} \neq \phi$ for $t > 0$, there exist some i $(0 \leqq i \leqq k)$ for which $a_{i0} \neq 0$. Because if not, then we have

$$\operatorname{ord}(F_{3\lambda(t)}) = \operatorname{ord}\left(u_t \cdot \prod_{i=-1}^{k} f_{it}\right) \geqq \deg(x) + \deg(y) + \sum_{i=0}^{k} \deg(f_{it})$$

$$= 1/c + 1/d \text{ for } (1/c,\ 1/d)$$

namely

$$\operatorname{ord}(F_{3\lambda(t)}) \geqq 1 \text{ for } (b/(ab + a + b),\ a/(ab + a + b)).$$

Let c_g be the minimum of the set $\{c_i | a_{i0} \neq 0\ (0 \leqq i \leqq k)\}$. Suppose that $a_{g0} = a_{i0}$ if $c_g = c_i$. Then if $c_g = c_0$, then $F_{3\lambda(t)}$ contains the terms

$$K \cdot xy^{(k-m)d}(y - a_{00}x^{c_g})^{md+1}$$

and if not, then $F_{3\lambda(t)}$ contains the terms

$$K \cdot xy^{(k-m)d+1}(y - a_{g0}x^{c_g})^{md}$$

since

$$\begin{cases} \operatorname{In}(f_{it}) = (y - a_{g0}x^{c_g})^d & \text{for } (1/c_g d,\ 1/d) \text{ if } c_g = c_i \\ \operatorname{In}(f_{it}) = y^d & \text{for } (1/c_g d,\ 1/d) \text{ otherwise} \end{cases}$$

and

$$\begin{cases} \mathrm{In}\,(f_{0t}) = y - a_{00}x^{c_g} & \text{for } (1/c_g d,\ 1/d) \text{ if } c_g = c_0 \\ \mathrm{In}\,(f_{0t}) = y & \text{for } (1/c_g d,\ 1/d) \text{ otherwise} \end{cases}$$

$$\mathrm{In}\,(u_t) = K \text{ (a constant) for any weights.}$$

Thus $F_{3\lambda(t)}$ contains the monomial $x^{c_g+1}y^b$ with non-zero coefficient. But this contradicts the fact that $F_{3\lambda(t)}$ does not contain the monomials $x^{i+1}y^b$ $(i \geqq 1)$ with non-zero coefficients. Hence if $\#\{i | c_g = c_i\ (0 \leqq i \leqq k)\} = 1$, then we have immediately a contradiction. If $\#\{i | c_g = c_i\ (0 \leqq i \leqq k)\} \geqq 2$, then we have a contradiction in the same discussion of the intersection number as in Lemma 2. Hence we have $\mathscr{S}_{F_3} - \mathscr{A}_{F_3} = \phi$, and we have $\mathscr{S}_{F_3} = \mathscr{A}_{F_3}$ since $\mathscr{S}_{F_3} \supset \mathscr{A}_{F_3}$. These complete the proof.

Now we are in a stage to show our theorem.

THEOREM. *Let f be a quasihomogeneous function of corank two with an isolated critical point. Then the stratum with constant Milnor number of a mini-transversal family of f equals a germ of non-singular algebraic subset at the origin in $\mathbf{C}^{\mu-1}$ of dimension $m(f)$, where $m(f)$ is the inner modality of f and μ is the Milnor number of f.*

Proof. Let f_0 be the principal partrt of f. Since f_0 has an isolated critical point, f_0 has one of the following pairs of monomials with non-zero coefficients (up to permutation of variables x, y); x^a, y^b $(a, b \geqq 3)$, x^{a+1}, $xy^b (a, b \geqq 2)$, $x^{a+1}y$, xy^{b+1} $(a, b \geqq 1)$. Thus we may assume that f_0 is one of the quasihomogeneous functions f_i $i = 1, 2, 3$. We put

$$G_i(x, y, z, t) = F_i(x, y, t) + z_1{}^2 + \cdots + z_{n-2}^2 \quad i = 1, 3$$

$$G_2(x, y, z, t) = \begin{cases} F_2(x, y, t) + z_1{}^2 + \cdots + z_{n-2}^2 \\ F_2'(x, y, t) + z_1{}^2 + \cdots + z_{n-2}^2 \end{cases}.$$

Then G_{jt} is a mini-transversal family of f for some j $(j = 1, 2, 3)$. Since $\mu(G_{jt}) = \mu(f)$ if and only if $\mu(F_{jt}) = \mu(f_j)$, by the preceeding lemmata we have

$$\mathscr{S}_{G_j} = \mathscr{A}_{G_j}.$$

Let G_s be any mini-transversal family of f and let $\tau: (\mathbf{C}_s^{\mu-1}, o) \to (\mathbf{C}_t^{\mu-1}, o)$ be a biholomorphic mapping under which the mini-transversal family G_s is equivalent to the mini-transversal family G_{jt}. By the mapping τ, we have the analytic isomorphism

$$\mathscr{S}_G \simeq \mathscr{S}_{G_j}.$$

Hence S_G is a germ of non-singular algebraic subset at the origin in $\mathbf{C}^{\mu-1}$ of the same dimension as $\mathscr{A}_{G_j}$. Since the dimension of $\mathscr{A}_{G_j}$ is equal to the inner modality of f, we have the conclusion.

We have the following corollary immediately from the definitions of modality and inner modality.

COROLLARY. *The modality of a quasihomogeneous function of corank two with an isolated critical point equals its inner modality.*

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